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HETEROGENEOUS EXPECTATIONS AND EQUILIBRIUM PRICE OF A RISKY ASSET: A NOTE

ABSTRACT

This paper presents a formal proof that the joint presence of heterogeneous expectations and restrictions on short sales of risky assets causes an upward bias in the risky asset's equilibrium price (as opposed to that which would have prevailed under no restriction on short sales). This upward price bias is an increasing function of the degree of heterogeneity in information set across investors. Our proof differs in that a general class of concave utility function is assumed rather than a specific form of utility function such as the constant absolute risk aversion utility function.

I. INTRODUCTION

The objective of this paper is to present a formal proof, without recourse to a specific form of utility function, that the joint presence of heterogeneous expectations and restrictions on short sales causes an upward bias in the risky asset's equilibrium price as opposed to that which would have prevailed under no restriction on short sales; and this upward bias increases when the degree of heterogeneity in information set across investors increases.

II. THEORY

IIA. Assumptions

The economy is described as follows:

1) There are two assets: one risky asset and one risk-free asset. For simple analysis, the price of risk-free asset is assumed to be always 1 (without loss of generality).[1]

2) The return on the risky asset follows a continuous-time stochastic (Wiener) process:

$$\frac{dP_t}{P_t} = \mu\, dt + \sigma\, dZ \tag{1}$$

where P_t is the price of the risky asset at time t; μ is the instantaneous expected rate of return for the risky asset; σ is the instantaneous standard deviation of the risky asset's rate of return; and dZ is a standard Wiener process.

3) Investors (denoted by subscript or superscript $k=1,\ldots,K$) are risk averse price takers, maximizing individual expected utility of lifetime consumption. They have heterogeneous expectations about μ, but have complete agreement on σ. This "partial" heterogeneous expectations assumption arises from the continuous-time return generating process. In other

words, investors observe the realized return "continuously," and consequently agree on the variability of the stochastic return (see Williams [6] and Merton [3]). However, investors can have different estimates for the risky asset's expected return; and μ^k is denoted as the k^{th} investor's subjective estimate for the risky asset's expected return.

4) In order to emphasize that short sales of risky assets are costly, it is assumed that information-motivated short sales of the risky asset are not allowed. The ability of investors to sell securities short without incurring transaction cost is severely limited by market practices such as escrow account and margine requirement.[2] Furthermore, traders on the floor are not even allowed to sell a security short unless the last transaction of the security at the exchange recorded a price increase, or "uptick." As would be expected by these institutional barriers, the short interest[3] of the New York Stock Exchange has been less than 0.3 percent of the total number of shares outstanding at the Exchange (N.Y.S.E. Fact Book).[4] Therefore, the assumption of no short sale should be acceptable as a first approximation to represent substantial transaction cost incurred by short sales of risky assets.

IIB. Optimal Consumption and Investment Decision

Given these assumptions, the investor's optimization system at time t, 0<t<T, is described as:

$$\max_{\{C_t^k, N_t^k\}} E_t^k\left[\int_t^T U_k(C_s^k, s)\,ds + V_k(W_T^k, T)\right] \tag{2}$$

subject to

$$dW_t^k = W_{t+dt}^k - W_t^k = N_t^k\, P_t(\mu^k\, dt + \sigma\, dZ) - C_t^k dt \tag{3}$$

$$N_t^k \geq 0 \tag{4}$$

where E_t^k is the conditional expectation operator; U_k is the utility function ($U_k' > 0$ and $U_k'' < 0$); C_t^k is consumption; N_t^k denotes the number of the risky asset; V_k is the bequest function which is also assumed to be strictly concave; W_t^k represents investor's wealth; and all other variables are as defined as before. For simplicity, all investors are assumed to face the same time horizon, T. In this optimization system, (3) represents the budget constraint, and (4) represents the restriction on short sales of the risky asset.

By the Bellman principle of optimality, the objective function is expressed as:

$$J^k(W_t^k,t) = \underset{\{C_t^k,N_t^k\}}{\mathrm{MAX}} \; E_t^k[U_k(C_t^k,t) + J^k(W_{t+dt}^k,t+dt)] \tag{5}$$

where $J^k(\cdot)$ is also strictly concave, and is referred to as the derived utility function of wealth; and (5) is also subject to the same constraints (3) and (4). By expanding J^k in the right hand side of (5) in a Taylor series about W_t^k and t, the Hamiton-Jacobi equation of dynamic optimality is derived to be:

$$\emptyset = \underset{\{C_t^k,N_t^k\}}{\mathrm{MAX}} \; \{U_k(C_t^k,t) + J_t^k + J_w^k\, E_t^k[dW_t^k] + \tfrac{1}{2} J_{ww}^k\, E_t^k[dW_t^k]^2\} \tag{6}$$

where $(dt)^n = \emptyset$ if $n \geq 3/2$; dt is suppressed for simple expression; J_t^k and J_w^k are first derivatives of J^k with respect to t and W_t^k, respectively; and J_{ww}^k is the second derivative of J^k with respect to W_t^k. Since (6) is subject to the constraints (3) and (4), the optimization system can be expressed as:

$$\emptyset = \underset{\{C_t^k, N_t^k\}}{MAX} \{U_k(C_t^k,t) + J_t^k + J_w^k [N_t^k P_t \mu^k - C_t^k] + \frac{1}{2} J_{ww}^k (N_t^k)^2 \sigma^2 P_t^2 + \lambda_t^k N_t^k\}$$

where λ^k is the Lagrangian multiplier for the short sale constraint. The first order conditions for (7) are:

$$\frac{\partial U_k(C^k,t)}{\partial C^k} = J_w^k > \emptyset \tag{8}$$

$$J_w^k P \mu^k + J_{ww}^k N^k P^2 \sigma^2 + \lambda^k = \emptyset \tag{9}$$

where subscript t is, hereafter, suppressed for simple notation. In (9), the Kuhn-Tucker condition should be held such that:

$$\begin{aligned} \lambda^k &= \emptyset \text{ if } N^k > \emptyset \\ \lambda^k &= -J^k P \mu^k > \emptyset \text{ if } N^k = \emptyset \end{aligned} \tag{10}$$

that is, λ^k is the shadow price of the short sale constraint, and the condition for a positive λ^k is that $\mu^k < \emptyset$.[5] Obviously, if investors are pessimistic about the risky asset, they do not hold the risky asset long. By (9) and (10), the optimal number of the risky asset, N^k, is:

$$N^k = \begin{cases} \dfrac{-J_w^k}{J_{ww}^k} \left(\dfrac{\mu^k}{\sigma^2}\right) \dfrac{1}{P} & \text{if } \mu^k > \emptyset \\ \emptyset & \text{if } \mu^k \leq \emptyset \end{cases} \tag{11}$$

IIC. <u>The Upward Bias in the Equilibrium Price of Risky Asset</u>

Let N be the total number of the risky asset. For the analysis of equilibrium price, investors are partitioned into two groups: the first K_0 investors (referred to as optimistic investors) hold the risky asset long and the remaining $K - K_0$ investors (referred to as pessimistic investors) do not hold the risky asset.

The market equilibrium condition is obtained by equating the aggregate demand and the aggregate supply of the risky asset; and by solving for P:

$$P = \frac{1}{N} \sum_{k=1}^{K_0} \theta^k \left(\frac{\mu^k}{\sigma^2}\right) \tag{12}$$

where $\theta^k = (-J_w^k / J_{ww}^k)$ is defined as the k^{th} investor's risk tolerance (i.e., the inverse of absolute risk aversion).

If there were no restriction on short sales of the risky asset, pessimistic investors' expectations would have been incorporated into the equilibrium price such that:

$$P^* = \frac{1}{N} \sum_{k=1}^{K} \theta^k \left(\frac{\mu^k}{\sigma^2}\right) \tag{13}$$

where P* is the equilibrium price of the risky asset under no restriction of short sales. Therefore, the price bias becomes:

$$P - P^* = \frac{-1}{N} \sum_{k=K_0+1}^{K} \theta^k \left(\frac{\mu^k}{\sigma^2}\right) > 0 \tag{14}$$

The price bias in (14) is positive because $\mu^k < 0$ for $k = K_0+1, \ldots, K$. This result conforms to the earlier works by Miller [4], Figlewski [1], and Jarrow [2]. However, this paper presents a formal proof without recourse to a specific form of utility function (e.g., the constant absolute risk aversion utility function by Figlewski and Jarrow).

IID. Upward Price Bias and Heterogeneity of Information

Suppose that the individual's information variable, Ω^k, is drawn from the normal distribution with mean, $\overline{\Omega}$, and variance, H^2. It is further assumed that μ^k is linear in Ω^k such that $\mu^k = \alpha + \beta\Omega^k$ $(\beta > 0)$. Then, μ^k

is also normally distributed with mean, $\alpha + \beta\bar{\Omega}$, and variance, $\beta^2 H^2$. Since $\mu^k < 0$ for pessimistic investors, the number of pessimistic investors, $K - K_0$, is:

$$K\, Z\left(\frac{-(\alpha+\beta\bar{\Omega})}{\beta H}\right) \tag{15}$$

where Z is the cumulative distribution function for the standardized normal random variable. For simplicity, let $\theta^k = \theta$ for all k. Then, the magnitude of the upward bias in the equilibrium price is expressed as:

$$P - P^* = \frac{-\theta K}{N}\, Z\left(\frac{-(\alpha+\beta\bar{\Omega})}{\beta H}\right) \left(\frac{\hat{\mu}}{\sigma^2}\right) \tag{16}$$

where $\hat{\mu}$ is the average of pessimistic investors' estimates for the risky asset's expected return. Equation (16) shows that the magnitude of the upward bias in the equilibrium price increases when the degree of heterogeneity in information set across investors, H, increases.

III. CONCLUSION

The valuation model developed in this paper has "formally" proved that the joint presence of heterogeneous expectations and restrictions of short sales results in an upward bias in the equilibrium price of risky assets; and the magnitude of this upward price bias increases when the degree of heterogeneity in information set across investors increases.

FOOTNOTES

[1]The assumptions of a single risky asset and the zero risk-free rate are inconsequential to the final result of this paper.

[2]In reality, brokers play the role of the security lender. The proceeds from short sales are held in the escrow account of the brokerage house. In addition, brokers require an additional amount (currently 50% of the short sales proceeds) to be deposited (margin requirement). See Rudd and Schroeder [5] for further discussion of transaction costs associated with short sales.

[3]The short interest refers to the number of shares sold short and not covered.

[4]In fact, most of these short sale transactions are generated by non-information (technical) motives of specialists and exchange members to meet the influx of public buy orders and to maintain "orderly" markets. Note that these "liquidity-motivated" short sales are not relevant to this paper's purpose.

[5]If the risk-free rate is r, it can be shown that the investor does not hold the risky asset if $\mu^k < r$.

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